SHENTON COLLEGE

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Examination Semester Two 2016
Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3 + 4

Section One
(Calculator-free)

Your name ______________________________

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section One.
Formula sheet.

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters
Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this examination

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of exam
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	12	12	100	98	65
			Total	150	100

Instructions to candidates

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Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: if you need to use the space to continue an answer, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than 2 marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

It is recommended that you **do not use pencil, except in diagrams**.

STRUCTURE OF THIS PAPER

QUESTION	MARKS AVAILABLE	MARKS AWARDED
1	5	
2	8	
3	5	
4	5	
5	8	
6	5	
7	8	
8	8	
TOTAL	**52**	

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(5 marks)**

The polynomial $h(z) = z^4 - 6z^3 + 3az^2 - 30z + 10a$, where a is a real constant, has a zero of $3 - i$. Determine the value of a and all other zeros of $h(z)$.

If $z-(3-i)$ is one factor, then $z-(3+i)$ must be another.

✓ complex conjugate

$$(z-(3-i))(z-(3+i))$$

$$= z^2 - (3+i)z - (3-i)z + (3+i)(3-i)$$

✓ split $h(z)$ into two quadratics

$$= z^2 - 6z + 10.$$

$$h(z) = (z^2-6z+10)(z^2+xz+a)$$

✓ compare coefficients

For the z^3 term,

✓ solve for a

$$-6 + x = -6$$

✓ list roots

$$\Rightarrow x = 0$$

For the z^2 term

$$10 - 6x + a = 3a$$

$$\Rightarrow a = 5$$

$$h(z) = (z^2-6z+10)(z^2+5).$$

The second bracket gives roots of $\sqrt{5}i$ and $-\sqrt{5}i$

So all 4 roots are $3-i$, $3+i$, $\sqrt{5}i$ & $-\sqrt{5}i$

Question 2 **(8 marks)**

Two functions are defined by $f(x) = \sqrt{3x - 1}$ and $g(x) = \frac{1}{x}$.

(a) Determine the composite function $f(g(x))$ and the domain over which it is defined. (3 marks)

$f(g(x)) = \sqrt{\frac{3}{x} - 1}$

Defined for $\frac{3}{x} - 1 \geq 0$

$\frac{3}{x} \geq 1$

$3 \geq x$

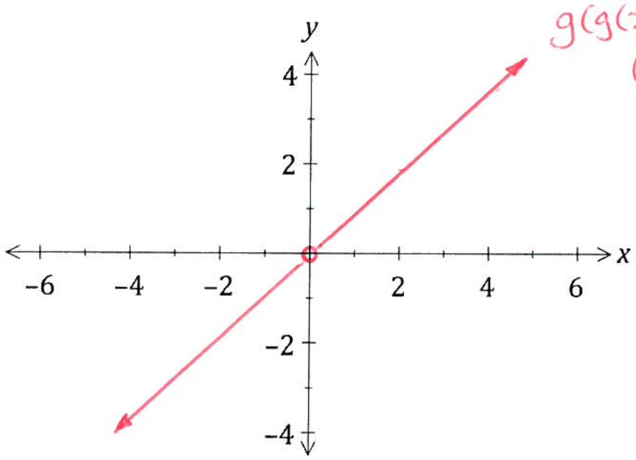
However, $\frac{3}{x} \geq 1 \Rightarrow x > 0$

$\therefore$ DOMAIN

$\{x : 0 < x \leq 3\}$

✓ composite
✓ $x \leq 3$
✓ $x > 0$

(b) Sketch the graph of $y = g(g(x))$ on the axes below. (2 marks)

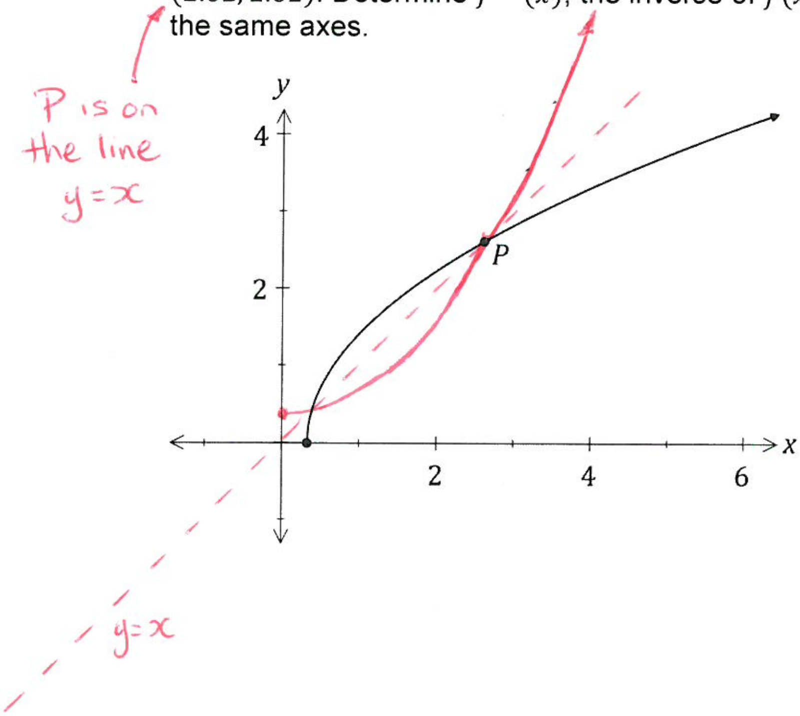


$g(g(x)) = x$
$(x \neq 0)$

✓ $y = x$
✓ excludes $x = 0$

(c) The graph of $y = f(x)$ is shown below, passing through point P with coordinates $(2.62, 2.62)$. Determine $f^{-1}(x)$, the inverse of $f(x)$, and sketch the graph of $y = f^{-1}(x)$ on the same axes. (3 marks)

P is on the line $y = x$



For inverse,

$x = \sqrt{3y - 1}$

$x^2 = 3y - 1$

$\frac{x^2 + 1}{3} = y$

✓ $f^{-1}(x)$
✓ points (through P)
✓ curve

Question 3 **(5 marks)**

An object, initially at rest, is dropped from the top of tall building so that after t seconds it has velocity v meters per second.

The air resistance encountered by the object is proportional to its velocity, so that the velocity satisfies the equation $\frac{dv}{dt} = 10 - kv$, where k is a constant.

(a) Express the velocity of the object in terms of t and k. (4 marks)

$$\int \frac{1}{10 - kv}\, dv = \int dt$$

$$-\frac{1}{k} \ln|10 - kv| = t + c$$

$$\ln|10 - kv| = -kt - kc$$

$$10 - kv = e^{-kc} e^{-kt}$$

when $t = 0,\ v = 0$

$$\Rightarrow e^{-kc} = 10$$

$$10 - kv = 10e^{-kt}$$

$$v = \frac{10 - 10e^{-kt}}{k}$$

✓ separate variables

✓ integrate

✓ use condition to determine c

✓ $v =$

(b) Sensors on the object indicate that its velocity will never exceed 55 metres per second. Determine the value of the constant k. (1 mark)

$$\lim_{t \to \infty} \frac{10 - 10e^{-kt}}{k} \leq 55$$

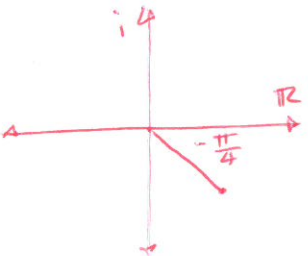
$$\Rightarrow \frac{10}{k} \leq 55$$

$$\Rightarrow \frac{10}{55} \leq k. \quad \text{or} \quad \frac{2}{11} \leq k$$

Question 4 **(5 marks)**

Let $v = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

(a) Express v in polar form. (2 marks)



$\theta = -\frac{\pi}{4}$

$r = 1$

$v = \text{cis}\left(-\frac{\pi}{4}\right)$

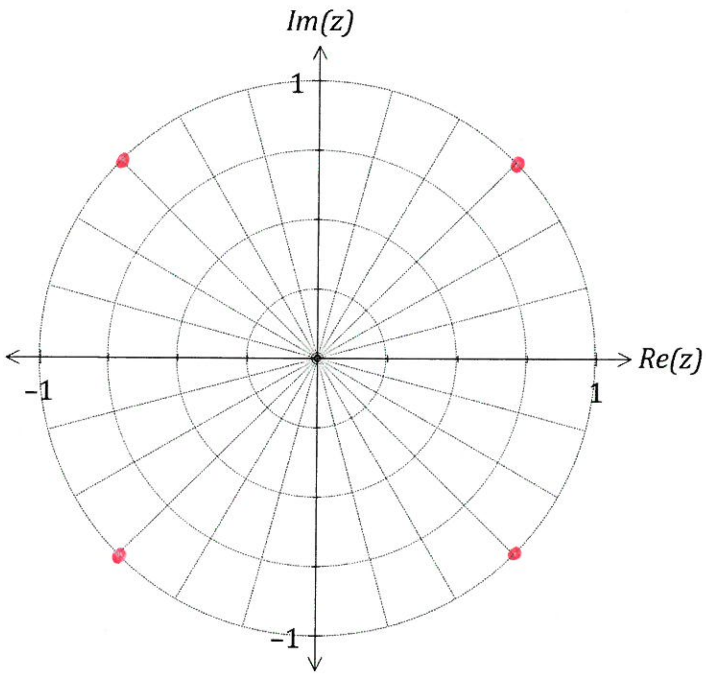
(b) Show that $v^4 = -1$. (1 mark)

$v^4 = \text{cis}\left(4 \times -\frac{\pi}{4}\right)$

$= \cos(-\pi) + i\sin(-\pi)$

$= -1$

(c) Plot the roots of $z^4 + 1 = 0$ on the following Argand diagram. (2 marks)



Question 5 **(8 marks)**

(a) Using partial fractions, or otherwise, determine $\int \frac{x-19}{(x+1)(x-4)}\,dx$. (4 marks)

$$\frac{x-19}{(x+1)(x-4)} = \frac{A}{x+1} + \frac{B}{x-4}$$

✓ splits binomial

✓ A

✓ B

✓ integrates

$$x-19 = Ax - 4A + Bx + B$$

$$\begin{cases} A+B = 1 \\ -4A+B = -19 \end{cases}$$

$$-4A + (1-A) = -19$$

$$A = 4$$

$$B = -3$$

$$\int \frac{4}{x+1} - \frac{3}{x-4}\,dx$$

$$= 4\ln|x+1| - 3\ln|x-4| + c$$

(b) Use the substitution $u = \sin x$ to evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$. (4 marks)

$u = \sin x$

$\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

when $x = \frac{\pi}{6}$,

$u = \frac{1}{2}$

when $x = \frac{\pi}{2}$

$u = 1$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{\sin x}}\,dx$$

$$= \int_{\frac{1}{2}}^{1} \frac{\cos x}{\sqrt{u}} \frac{du}{\cos x}$$

$$= \int_{\frac{1}{2}}^{1} \frac{1}{\sqrt{u}}\,du$$

$$= 2\sqrt{u}\Big]_{\frac{1}{2}}^{1}$$

$$= 2 - \frac{2}{\sqrt{2}}$$

$$= 2 - \sqrt{2}$$

✓ dx

✓ bounds

✓ integrate

✓ evaluate.

Question 6 **(5 marks)**

The differential equation $y' = \frac{2x}{y}$ is shown in just one of the four slope fields below.

$\frac{dy}{dx} = \frac{2x}{y}$

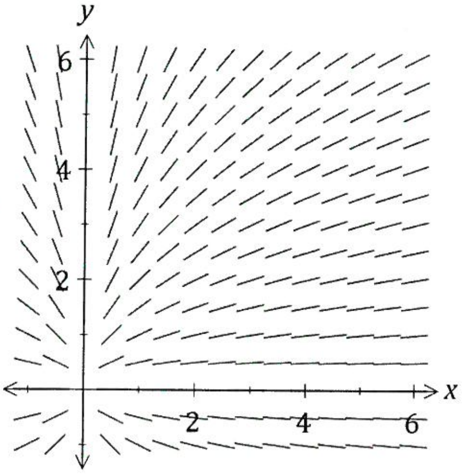
$\int y\,dy = \int 2x\,dx$

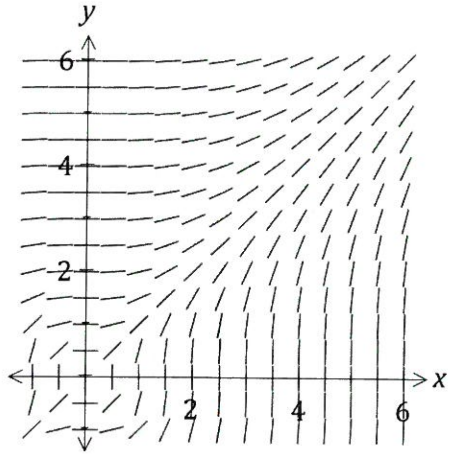
$\frac{y^2}{2} = x^2 + c$

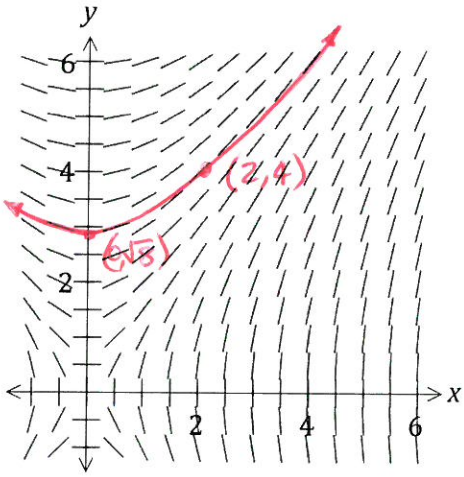
$y = \sqrt{2x^2 + c}$

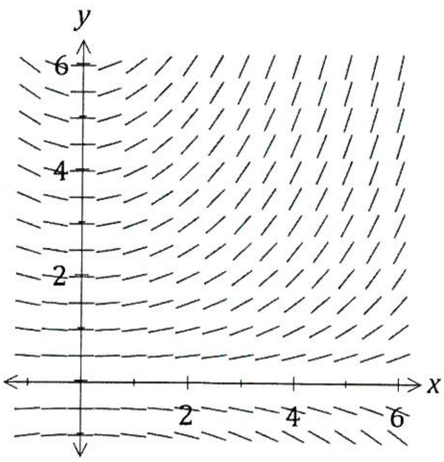
@ $(2,4)$

$y = \sqrt{2x^2 + 8}$









(a) On the slope field for $y' = \frac{2x}{y}$, sketch the solution of the differential equation that passes through the point $(2, 4)$. (3 marks)

✓ bottom left field
✓ slope
✓ includes $(2,4)$

(b) Another solution to the differential equation passes through the point $(6, -3)$. Use the incremental formula $\delta y \approx \frac{dy}{dx} \times \delta x$, with $\delta x = \frac{1}{10}$, to estimate the y-coordinate of this curve when $x = 6.1$. (2 marks)

@ $(6, -3)$, $\frac{2x}{y} = -4$

$\delta y = -4 \times \frac{1}{10}$

$= -0.4$

$\therefore$ y coord ≈ -3.4

✓ $\frac{dy}{dx}$
✓ y

Question 7 **(8 marks)**

The function f is defined as $f(x) = \frac{x^2-1}{x^2+1}$.

(a) Show that the **only** stationary point of the function occurs when $x = 0$. (2 marks)

$$f'(x) = \frac{2x(x^2+1) - 2x(x^2-1)}{(x^2+1)^2}$$

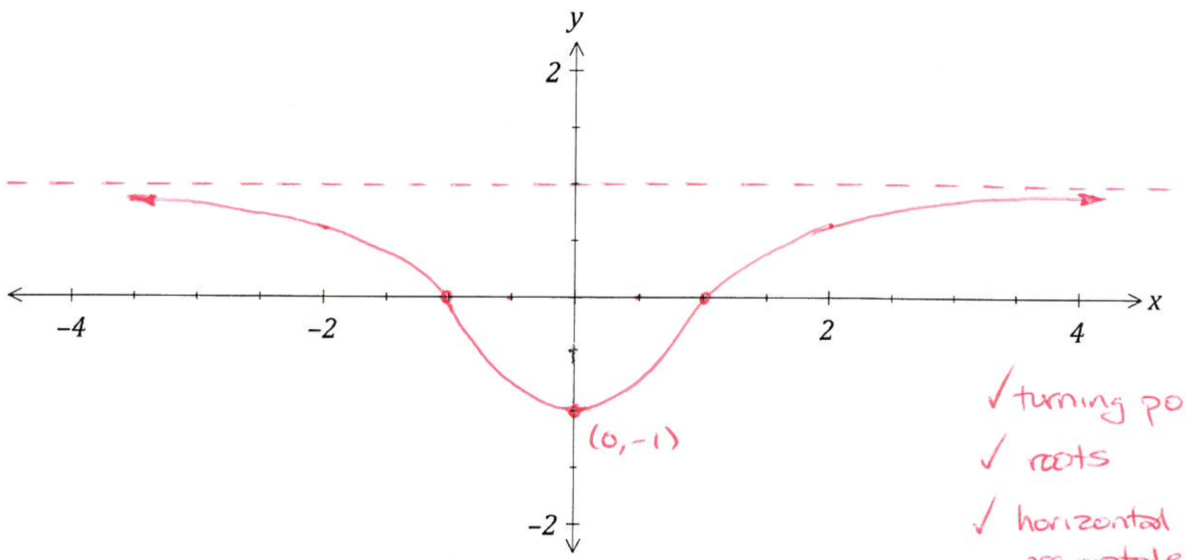
$$= \frac{4x}{(x^2+1)^2}$$

$\therefore f'(x) = 0$ iff $x = 0$

✓ derivative

✓ statement.

(b) Sketch the graph of $y = f(x)$ on the axes below. (3 marks)



✓ turning point

✓ roots

✓ horizontal assymptote.

(c) Using your graph, or otherwise, determine all solutions to

(i) $f(x) = |f(x)|$. (1 mark)

$x \in \mathbb{R} \setminus -1 < x < 1$

(ii) $f(x) = f(|x|)$. (1 mark)

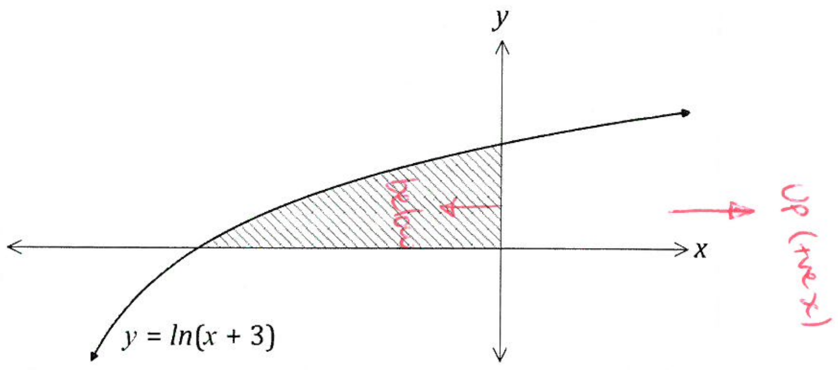
$x \in \mathbb{R}$

(iii) $f(x) = \frac{1}{f(x)}$. (1 mark)

$x = 0$

Question 8 **(8 marks)**

A region is bounded by $x = 0, y = 0$ and $y = \ln(x + 3)$ as shown in the graph below.



(a) Show that the area of the region is given by $\int_0^{\ln 3}(3 - e^y)\,dy$. (3 marks)
(Do not evaluate the integral).

$y = \ln|x+3|$

$e^y = x+3$

$e^y - 3 = x$

$x = 0 \Rightarrow y = \ln 3$

Region is "below" y-axis (when viewed sideways), so integral will be negative. Thus calculate for

$-(e^y - 3) = 3 - e^y$

✓ rearrange for $x =$

✓ bound

✓ sign

(b) Determine the volume of the solid generated when the region is rotated through 2π about the y-axis. (5 marks)

$$\text{Volume} = \int_0^{\ln 3} \pi (e^y - 3)^2\,dy$$

$$= \pi \int_0^{\ln 3} (e^{2y} - 6e^y + 9)\,dy$$

$$= \pi \left[\frac{e^{2y}}{2} - 6e^y + 9y\right]_0^{\ln 3}$$

$$= \pi \left[\left(\frac{9}{2} - 18 + 9\ln 3\right) - \left(\frac{1}{2} - 6\right)\right]$$

$$= \pi(-8 + 9\ln 3)$$

✓ formula

✓ expand

✓ integrate

✓ substitute

✓ evaluate

End of questions

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Examination Semester Two 2016
Question/Answer Booklet

MATHEMATICS SPECIALIST UNIT 3 + 4

Section Two
(Calculator-assumed)

Your name _______________________________

Time allowed for this section

Reading time before commencing work: 10 minutes
Working time for paper: 100 minutes

Material required/recommended for this section

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Formula Sheet (retained from Section One)

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in the WACE examinations

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Structure of this examination

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STRUCTURE OF THIS PAPER

QUESTION	MARKS AVAILABLE	MARKS AWARDED
9	6	
10	8	
11	7	
12	13	
13	10	
14	7	
15	8	
16	8	
17	7	
18	7	
19	10	
20	7	
TOTAL	**98**	

Section Two: Calculator-assumed 65% (97 Marks)

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 9 (6 marks)

A system of equations is shown below.

$$\begin{aligned} x + 2y + 3z &= 1 \\ y + 3z &= -1 \\ -y + (a^2 - 4)z &= a + 2 \end{aligned}$$

(a) Determine the unique solution to the system when $a = 2$. (2 marks)

From bottom equation, $y = -4$

Moving upwards, $z = 1$

$x = 6$

$\therefore$ Unique solution for $x = 6, y = -4, z = 1$

✓ solve for y

✓ completes solution

(b) Determine the value(s) of a so that the system

(i) has an infinite number of solutions. (3 marks)

Adding rows 2 & 3 gives

$(a^2 - 4 + 3)z = a + 1$

$(a^2 - 1)z = a + 1$

$(a+1)(a-1)z = a + 1$

When $a = -1$, $0z = 0$

$\Rightarrow$ infinite solutions

✓ uses multiples of rows to generate equations

✓ factorise a

✓ determine appropriate solution

(ii) has no solutions. (1 mark)

From (i), when $a = 1$,

$0z = 2$

$\Rightarrow$ no solutions

Question 10 **(8 marks)**

The length of time, T months, that an athlete stays in an elite squad can be modelled by a normal distribution with population mean μ and population variance $\sigma^2 = 15$.

(a) An independent sample of five values of T is $7.7, 15.2, 3.9, 13.4$ and 11.8 months.

(i) Calculate the mean of this sample and state the distribution that a large number of such samples is expected to follow. (2 marks)

$\bar{x} = 10.4$ months.

✓ $\bar{x} = 10.4$

✓ $N(\mu, 3)$

Sample means will be normally distributed

$N(\mu, \frac{15}{5})$

(ii) Use this sample to construct a 90% confidence interval for μ, giving the bounds of the interval to two decimal places. (3 marks)

$z = 1.645$

✓ z

Interval $10.4 \pm 1.645 \times \sqrt{3}$

✓ use of $\sqrt{3}$

$= (7.55, 13.25)$

✓ bounds

(b) Determine the smallest number of values of T that would be required in a sample for the total width of a 95% confidence interval for μ to be less than 3 months. (3 marks)

For total width 3, $z\frac{\sigma}{\sqrt{n}} = 1.5$

ie $1.96 \times \frac{\sqrt{15}}{\sqrt{n}} = 1.5$

✓ use of 1.5

✓ n

$n = \left(\frac{1.96 \times \sqrt{15}}{1.5}\right)^2$

✓ rounding & conclusion

$= 25.61$

Sample must be at least 26

Question 11 **(7 marks)**

Plane p_1 has equation $3x + y + z = 6$ and line l has equation $\mathbf{r} = \mathbf{i} + \mathbf{j} + 2\mathbf{k} + t(\mathbf{i} - 2\mathbf{j} - \mathbf{k})$.

(a) Show that the line l lies in the plane p_1. (3 marks)

$p_1 \rightarrow \tilde{r} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 6 \qquad l \rightarrow \tilde{r} = \begin{pmatrix} 1+t \\ 1-2t \\ 2-t \end{pmatrix}$.

✓ equations in vector form

intersecting the two: $\begin{pmatrix} 1+t \\ 1-2t \\ 2-t \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = 6$

✓ intersects

✓ interpret

$3 + 3t + 1 - 2t + 2 - t = 6$

$6 = 6$

This is true $\forall t$, therefore every point on the line intersects the plane, ie the line lies in the plane.

(b) Another plane, p_2, is perpendicular to plane p_1, parallel to the line l and contains the point with position vector $i - 3j - k$. Determine the equation of plane p_2, giving your answer in the form $ax + by + cz = d$. (4 marks)

$\perp$ to $p_1 \Rightarrow p_2$ must contain $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

$/\!/$ to $l \Rightarrow p_2$ must contain $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$.

✓ identify two vectors

$\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix}$

✓ cross product

✓ k.

✓ cartesian form.

$p_2 \rightarrow \tilde{r} \cdot \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix} = k$.

$\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix} = -4$.

$p_2 \rightarrow \tilde{r} \cdot \begin{pmatrix} 1 \\ 4 \\ -7 \end{pmatrix} = -4$

or $x + 4y - 7z = -4$

Question 12 **(13 marks)**

(a) Show that the gradient of the curve $2x^2 + y^2 = 3xy$ at the point $(1, 2)$ is 2. (3 marks)

Differentiating implicitly,

$$4x + 2y\frac{dy}{dx} = 3y + 3x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{3y - 4x}{2y - 3x}$$

@ $(1,2)$, $\frac{dy}{dx} = \frac{6-4}{4-3}$

$= 2$

✓ implicit
✓ product
✓ substitute to show

(b) Another curve passing through the point $(-2, 10)$ has gradient given by $\frac{dy}{dx} = \frac{2xy}{1+x^2}$. Use a method involving separation of variables and integration to determine the equation of the curve. (4 marks)

$$\int \frac{1}{y}\,dy = \int \frac{2x}{1+x^2}\,dx$$

$$\ln y = \ln|1+x^2| + c$$

$$y = k(1+x^2)$$

$$10 = k(1+2^2)$$

$$k = 2$$

$$y = 2(1+x^2)$$

✓ separate variables
✓ integrate
✓ remove logs
✓ find constant

(c) A particle is moving along the curve given by $y = \sqrt[3]{x}$, with one unit on both axes equal to one centimetre. When $x = 1$, the y-coordinate of the position of the particle is increasing at the rate of 2 centimetres per second.

(i) Show that the x-coordinate is increasing at 6 centimetres per second at this instant. (2 marks)

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

✓ $\frac{dy}{dx}$

$$\frac{dx}{dt} = \frac{dx}{dy}\frac{dy}{dt}$$

✓ chain rule

$$= 3(1)^{\frac{2}{3}} \times 2$$

$$= 6$$

(ii) Determine the exact rate at which the distance of the particle from the origin is changing at this instant. (4 marks)

Let s be the distance from the origin

$$s = \sqrt{x^2 + y^2}$$

$$= \sqrt{x^2 + x^{\frac{2}{3}}}$$

✓ equation for distance

$$\frac{ds}{dx} = \frac{3x^{\frac{4}{3}} + 1}{3x^{\frac{1}{3}}\left(x^2 + x^{\frac{2}{3}}\right)^{\frac{1}{2}}}$$

✓ $\frac{ds}{dx}$ (do not need to show)

✓ chain rule

@ $x = 1$, $\frac{ds}{dx} = \frac{2\sqrt{2}}{3}$

✓ exact solution

$$\frac{ds}{dt} = \frac{ds}{dx}\frac{dx}{dt}$$

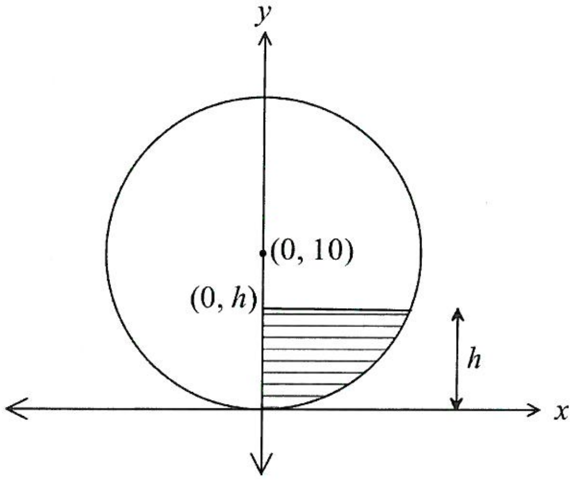
$$= \frac{2\sqrt{2}}{3} \times 6$$

$$= 4\sqrt{2}$$

Question 13 **(10 marks)**

(a) The circle with equation $x^2 + (y-10)^2 = 100$ is shown in the diagram below. (3 marks)

The shaded region of the circle is rotated about the y axis.



Show that the volume generated by the rotation is $V = \pi(10h^2 - \frac{h^3}{3})$.

$x^2 = 100 - (y-10)^2$

$= 20y - y^2$

$V = \pi \int_0^h x^2 \, dy$

$= \pi \int_0^h (20y - y^2) \, dy$

$= \pi \left[10y^2 - \frac{y^3}{3} \right]_0^h$

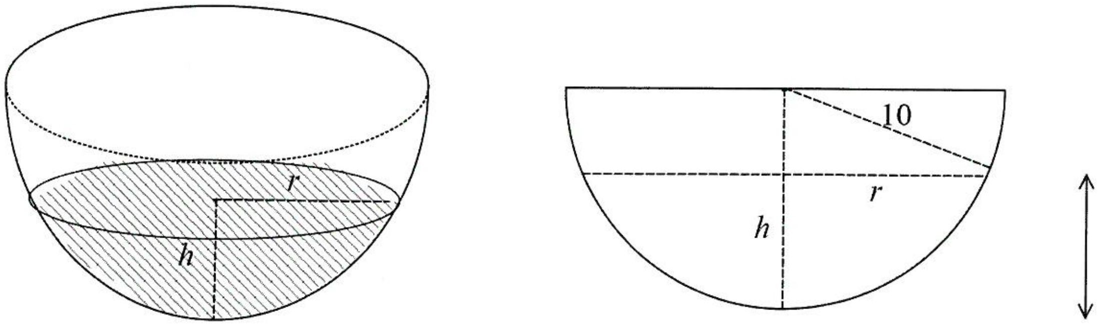
$= \pi \left(10h^2 - \frac{h^3}{3} \right)$

✓ formula

✓ integrate

✓ substitute

(b) A hemi-spherical vase of radius 10 cm is being filled with water at a rate of 10 cm^3 per second.



(i) Find the relationship between h and r. (1 mark)

$$r^2 = 20h - h^2$$

$$r = \sqrt{20h - h^2}$$

(ii) Use a calculus method to find the change in the radius r if the height h is increased from 8 cm to 8.1 cm. (3 marks)

when $h = 8$
$r = \sqrt{96}$

$$\delta r = \frac{dr}{dh}\,\delta h$$

✓ $r = \sqrt{96}$

$$2r\frac{dr}{dh} = 20 - 2h$$

✓ $\frac{dr}{dh}$

$$\left.\frac{dr}{dh}\right|_{(8,\sqrt{96})} = \frac{4}{2\sqrt{96}}$$

✓ δr

$$\delta r = \frac{0.4}{2\sqrt{96}} \simeq 0.02\text{ cm}$$

(iii) Find $\frac{dh}{dt}$ when $r = 8$ cm given $V = 10h^2 - \frac{h^3}{3}$. (3 marks)

$$\frac{dV}{dt} = \frac{dV}{dh}\,\frac{dh}{dt}.$$

✓ $\frac{dV}{dh}$

$$10 = (20h - h^2)\frac{dh}{dt}.$$

✓ $h = 4$

when $r = 8$, $h = 4$ (reject 16, $0 \le h \le 10$) ✓ chain rule

$$\frac{dh}{dt} = \frac{10}{64}$$

$$= \frac{5}{32}\text{ cm}$$

Question 14 **(7 marks)**

(a) The equation of a sphere with centre at $(2, -3, 1)$ is $x^2 + y^2 + z^2 = ax + by + cz - 2$.

Determine the values of a, b, c and the radius of the circle. (3 marks)

$$(x-2)^2 + (y+3)^2 + (z-1)^2 = r^2$$

$$x^2 - 4x + 4 + y^2 + 6y + 9 + z^2 - 2z + 1 = r^2$$

$$x^2 + y^2 + z^2 = 4x - 6y + 2z - 14 + r^2$$

$\therefore$ $a = 4,\ b = -6,\ c = 2,$

$r^2 - 14 = -2$

$\Rightarrow r = 2\sqrt{3}$

✓ use of centre formula

✓ a, b, c

✓ r

(b) Two particles, P and Q, leave their initial positions at the same time and travel with constant velocities shown in the table below.

Particle	Initial position	Velocity
P	$10\mathbf{i} - 5\mathbf{j} + 5\mathbf{k}$	$6\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$
Q	$28\mathbf{i} + 22\mathbf{j} - 31\mathbf{k}$	$2\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$

Show that the two particles collide, stating the position vector of the point of collision. (4 marks)

$$r_P = \begin{pmatrix} 10+6t \\ -5+2t \\ 5-4t \end{pmatrix} \qquad r_Q = \begin{pmatrix} 28+2t \\ 22-4t \\ -31+4t \end{pmatrix}$$

Solving one coordinate,

$10 + 6t = 28 + 2t$

$t = \frac{9}{2}$

@ $t = \frac{9}{2}$ $\quad r_P = \begin{pmatrix} 37 \\ 4 \\ -13 \end{pmatrix}, \quad r_Q = \begin{pmatrix} 37 \\ 4 \\ -13 \end{pmatrix}$

$\therefore$ with all coordinates equal, the particles collide at $\begin{pmatrix} 37 \\ 4 \\ -13 \end{pmatrix}$ when $t = \frac{9}{2}$.

✓ vector equations

✓ solve one coord.

✓ confirm other two

✓ position

Question 15 **(8 marks)**

(a) Briefly describe a reason that a sample rather than a complete population may be used when carrying out a statistical investigation. (1 mark)

It provides a whole topic that can be studied in the Year 12 course.

(b) A researcher used government records to select a random sample of the ages of 114 men who had died recently in a town close to an industrial complex. The mean and standard deviation of the ages in the sample were 73.3 and 8.27 years respectively.

(i) Explain why the sample standard deviation is a reasonable estimate for the population standard deviation in this case. (1 mark)

The sample is large and random.

(ii) Calculate a 98% confidence interval for the population mean and explain what the interval shows. (4 marks)

$z \quad 2.326$

$73.3 \pm 2.326 \times \frac{8.27}{\sqrt{114}} \rightarrow (71.5, 75.1)$

We can be 98% certain that the true population mean lies between 71.5 and 75.1 years.

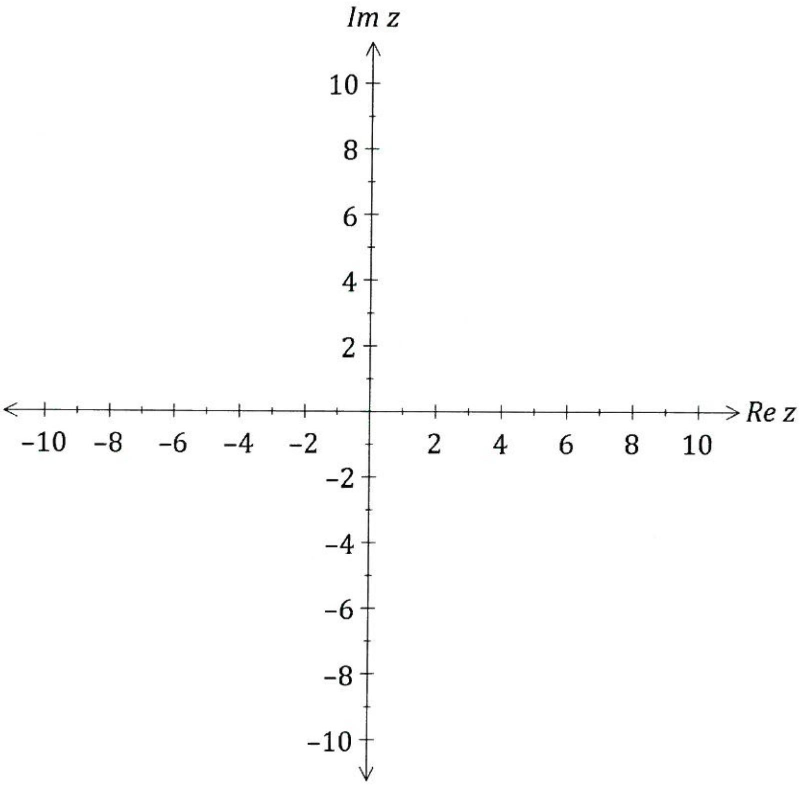
✓ z
✓✓ bounds
✓ statement

(iii) The national average life-span of men was known to be 75.3 years. State with a reason what conclusion the researcher could draw from the confidence interval calculated in (ii) about the life-span of men in the town. (2 marks)

As 75.3 is outside of the 98% interval, it is reasonable to say that there is a significant difference between the lifespan of the men in this town & the general population.

Question 16 **(8 marks)**

(a) On the Argand diagram below, clearly show the region that satisfies the complex inequalities given by $|z + 3 - 4i| \leq 5$ and $\frac{\pi}{2} \leq \arg z \leq \pi$. (4 marks)



✓ circle
✓ location
✓ shade inside
✓ Q2 only

(b) Determine all roots of the equation $z^5 = 16\sqrt{3} + 16i$, expressing them in the form $r \operatorname{cis} \theta$, where $r \geq 0$ and $-\pi \leq \theta \leq \pi$. (4 marks)

$z^5 = 32 \operatorname{cis} \frac{\pi}{6}$

$= r^5 \operatorname{cis} 5\theta$

$r = 2, \quad \theta = \frac{\pi}{30}$

$z_1 = 2 \operatorname{cis} \frac{\pi}{30}$

$z_2 = 2 \operatorname{cis} \frac{13\pi}{30}$

$z_3 = 2 \operatorname{cis} \frac{25\pi}{30}$

$z_4 = 2 \operatorname{cis} \frac{-11\pi}{30}$

$z_5 = 2 \operatorname{cis} \frac{-23\pi}{30}$

✓ polar
✓ r & θ
✓ all answers in domain
✓ full set through rotation.

Question 17 **(7 marks)**

(a) A market research company conducted a study on whether the packaging of breakfast cereals had an effect on the perceived healthiness of the product. Samples of 40 shoppers were shown three different boxes of cereal (which all had identical ingredients) and asked to rank them from 1 – 10 based on how healthy they thought they were. The first box, which was printed in colour and featured cartoon brand mascots, scored an average of 7.5. The second box, also in colour but showing scenes of a fruit orchard, scored 7.8. The third box, which was made of unbleached brown card and featuring simple icons representing wheat and nuts, scored 8.5. The standard deviations of all three samples was very close to 2.4.

(i) Determine the 90% confidence intervals for each of the three boxes. (3 marks)

Box 1 (6.876, 8.124)

Box 2 (7.176, 8.424)

Box 3 (7.876, 9.124)

(ii) Comment on the statistical significance of the effect of the packaging, based on your 90% confidence levels. (2 marks)

Between the ranges of 7.876 & 8.124 there is overlap of all 3 intervals, so there is no statistical significance of the effect of packaging at the 90% confidence level.

(b) Initial samples taken to determine the growth rates of farmed Murray cod measured 100 adult fish and found that the weights of the fish were normally distributed with a mean of 745g and a standard deviation of 50g.
A new sample is to be taken for which the investigators wish to be 95% confident that the mean weight of this second sample is within 7g of the population mean.
Assuming that the sample standard deviation is an accurate predictor of the population standard deviation, what should they use as the size of this second sample. (2 marks)

For 90%, $z = 1.960$ $\quad \varepsilon = 7$

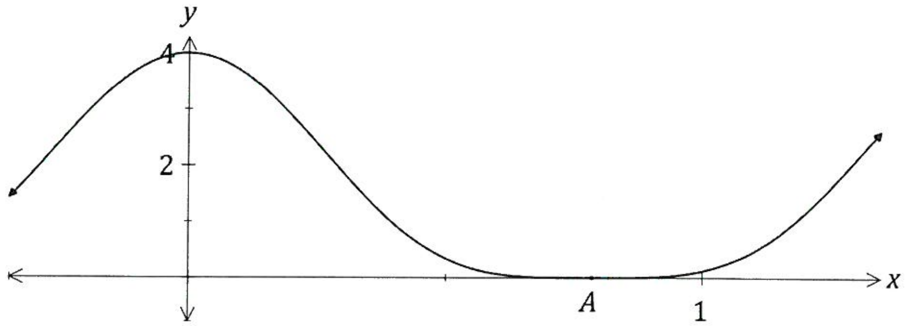
$$1.960 \times \frac{50}{\sqrt{n}} = 7$$

$$n = 196$$

The second sample must have at least 196 cod.

Question 18 **(7 marks)**

The graph of $y = f(x)$ is shown below, where $f(x) = 4\cos^4(2x)$ and A is the smallest root of $f(x),\ x > 0$.



(a) Show that $4\cos^4(2x) = \frac{3+4\cos(4x)+\cos(8x)}{2}$. (3 marks)

$$4\cos^4 2x = 4(\cos^2 2x)^2$$
$$= 4\left(\frac{1+\cos 4x}{2}\right)^2$$
$$= 1 + 2\cos 4x + \cos^2 4x$$
$$= 1 + 2\cos 4x + \frac{\cos 8x + 1}{2}$$
$$= \frac{3 + 4\cos 4x + \cos 8x}{2}$$

✓ factorise
✓ double angle
✓ simplify

(b) Hence determine $\int 4\cos^4(2x)\,dx$. (2 marks)

$$= \int \frac{3 + 4\cos 4x + \cos 8x}{2}\,dx$$
$$= \frac{3x}{2} + \frac{\sin 4x}{2} + \frac{\sin 8x}{16} + c$$

(c) Determine the exact volume of the solid generated when the region bounded by $y = f(x)$, $y = 0$, $x = 0$ and $x = A$ is rotated through 360° about the x-axis.

(2 marks)

$$\int_0^{\frac{\pi}{4}} \pi\left(4\cos^4 2x\right)^2 dx$$
$$= \frac{35\pi^2}{32}$$

Question 19 **(10 marks)**

(a) A small object has initial position vector $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} - \mathbf{k}$ metres and moves with velocity vector given by $\mathbf{v}(t) = 2t\mathbf{i} - 4t\mathbf{j} + 3\mathbf{k}$ ms^{-1}, where t is the time in seconds.

(i) Show that the acceleration of the object is constant and state the magnitude of the acceleration. (2 marks)

$\tilde{a}(t) = 2i - 4j$ ie constant acceleration

$|\tilde{a}(t)| = \sqrt{2^2 + 4^2}$

$= 2\sqrt{5}$

(ii) Determine the position vector of the object after 2 seconds. (3 marks)

$\tilde{r}(t) = t^2 i - 2t^2 j + 3t k + \tilde{c}$

$\tilde{r}(0) = i + 3j - k$

$\Rightarrow \tilde{c} = i + 3j - k$

$\tilde{r}(t) = (t^2+1)i + (3-2t^2)j + (3t-1)k.$

$\tilde{r}(2) = 5i - 5j + 5k.$

(b) Another small object has position vector given by $\mathbf{r}(t) = (1 + 2\sec t)\mathbf{i} + (3\tan t - 2)\mathbf{j}$ m, where t is the time in seconds.

(i) Determine the distance of the object from the origin when $t = \frac{\pi}{3}$. (2 marks)

$\tilde{r}(\frac{\pi}{3}) = (1 + 2\sec\frac{\pi}{3})i + (3\tan\frac{\pi}{3} - 2)j$

$|\tilde{r}(\frac{\pi}{3})| = 5.93$ (2dp)

(ii) Derive the Cartesian equation of the path of this object. (3 marks)

$x = 1 + 2\sec t$

$\sec t = \frac{x-1}{2}$

$y = 3\tan t - 2$

$\frac{y+2}{3} = \tan t.$

$\sec^2 t - \tan^2 t = 1$

$\Rightarrow \left(\frac{x-1}{2}\right)^2 - \left(\frac{y+2}{3}\right)^2 = 1$

✓ x & y

✓ trig identity

✓ substitute.

Question 20 **(7 marks)**

(a) A particle undergoing simple harmonic motion with a period of 5 seconds is observed to move in a straight line, oscillating 3.6 m either side of a central position. Determine the speed of the particle when it is 3 m from the central position. (3 marks)

$x(t) = 3.6\cos\left(\frac{2\pi}{5}t\right)$ ✓ $x(t)$

$3 = 3.6\cos\frac{2\pi}{5}t$ ✓ t

$t = 4.6$ sec. ✓ $|v(t)|$

$v(t) = -\frac{7.2\pi}{5}\sin\left(\frac{2\pi}{5}t\right)$

$|v(4.6)| = 2.5\,ms^{-1}$

(b) Another particle moving in a straight line experiences an acceleration of $x + 2.5$ ms^{-2}, where x is the position of the particle at time t seconds.

Given that when $x = 1$, the particle had a velocity of 2 ms^{-1}, determine the velocity of the particle when $x = 2$. (4 marks)

$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt}$

$a = \frac{dv}{dx}v$

$x+2.5 = \frac{dv}{dx}v$

$\int x+2.5\,dx = \int v\,dv$

$\frac{x^2}{2} + 2.5x + c = \frac{v^2}{2}$

$x^2 + 5x + c = v^2$

@ $x=1, v=2$

$\Rightarrow c = -2$

$x^2 + 5x - 2 = v^2$

when $x = 2$

$v^2 = 12$

$v = 2\sqrt{3}$

End of questions